A note on completely regular semigroups

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Let S be a semigroup. It is well known,[1]) that S is regular if and only if the relation

$$R \cap L = RL \tag{1}$$

holds for every left ideal L and for every right ideal R of S. It is natural to ask the following question: In what semigroups does a similar relation hold only for left or right ideals, respectively?

We shall prove in this short note, that a semigroup S satisfying the relation

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for each pair of left ideals in S, is a left regular semigroup, that is $a \in Sa^2$ for all a in S. Analogously, if a semigroup S satisfies the relation

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

for every pair of right ideals of S, then S is a right regular semigroup, that is $a \in a^2 S$, for each a in S. It will be also proved that the semigroup S is a union of disjoint groups provided it satisfies both (2) and (3) for left and right ideals, respectively. Finally, we give a characterization of semigroups having either property (2) for left ideals or property (3) for right ideals.

First we prove the following simple

Lemma. *In an arbitrary semigroup S*

$$(a)_L^2 = (a)a \quad \text{and} \quad (a)_R^2 = a(a), \tag{4}$$

where $a \in S$ and $(a)_L$ $[(a)_R, (a)]$ denotes the principal left [right, two-sided] ideal of S generated by a.

Proof. We have

$$(a)a = a^2 \cup aSa \cup Sa^2 \cup SaSa,$$

and

$$(a)_L^2 = (a \cup Sa)(a \cup Sa) = a^2 \cup a\,Sa \cup Sa^2 \cup SaSa$$

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[1]) See the references [1], [2] or [3]. Concerning the definitions of the fundamental notions in the algebraic theory of semigroups, we refer to the books [1] and [3].

because $(a)_L = a \cup Sa$ and $(a) = a \cup aS \cup Sa \cup SaS$. Therefore $(a)_L^2 = (a)a$, as we stated. The second statement is the left-right dual of the first one.

Theorem 1. *Let S be a semigroup satisfying the relation*

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for any two left ideals L_1, L_2 of S. Then every left ideal of S is a two-sided ideal of S.

Proof. Let $L_2 = S$, then relation (2) implies $L_1 = L_1 S$, therefore the left ideal L_1 is also a right ideal of S, which proves the theorem.

It is also true the following left-right dual of Theorem 1.

Theorem 2. *Let S be a semigroup satisfying the relation*

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

for any two right ideals R_1, R_2 of S. Then every right ideal of S is a two-sided ideal of S.

Theorem 3. *Let S be a semigroup satisfying relation* (2) *for any two left ideals. Then for any element a in S there exists at least one element x in S such that $a = xa^2$, i.e. S is a left regular semigroup.*

Proof. Let $L_1 = L_2 = (a)_L$. Then by (2) we have

$$(a)_L = (a)_L^2. \tag{5}$$

Applying the Lemma it follows that

$$(a)_L = (a)_L a, \tag{6}$$

because $(a)_L = (a)$ in view of Theorem 1. (6) implies

$$a \in (a)_L = (a \cup Sa)a = a^2 \cup Sa^2. \tag{7}$$

Thus we obtain either $a = a^2$ or $a = xa^2$, where $x \in S$. This means that $a \in Sa^2$ for any element a in S, i. e. the semigroup S is left regular. Theorem 3 is proved.

The dual statement reads as follows.

Theorem 4. *Let S be a semigroup satisfying relation* (3) *for any two right ideals of S. Then for each element a in S, there exists at least one element x in S so that $a = a^2x$, i.e. S is a right regular semigroup.*

Remark. The following example shows that converse of Theorem 3 is not true, i. e. relation (2) does not characterize the class of left regular semigroups:

	a	b	c	d
a	a	a	c	c
b	a	b	c	d
c	a	a	c	c
d	a	b	c	d

The semigroup $S = \{a, b, c, d\}$ with the above multiplication table is left regular, because S is an idempotent semigroup, but the relation (2) does not holds for any two left ideals of S. If

$$L_1 = \{a, b, c\}$$

and

$$L_2 = \{a, c, d\}$$

then

$$L_1 \cap L_2 = \{a, c\} \neq L_2 = L_1 L_2.$$

Now we prove the main result of this note.

Theorem 5. *If S is a semigroup having the property*

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for any two left ideals L_1, L_2 of S, and

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

for any two right ideals R_1, R_2 of S, then the semigroup S is a union of disjoint groups.[2])

Proof. Let S be a semigroup satisfying both the relations (2) and (3) for left and right ideals, respectively. Let a be an arbitrary element of S. Then by Theorem 1

$$(a)_L = (a) = (a)_R. \tag{8}$$

On the other hand $a = xa^2 = a^2 y$, where $x, y \in S$, by Theorems 3 and 4. Hence it follows that

$$Sa = (a) = aS. \tag{9}$$

We define an $\mathscr{L}$-equivalence in S as follows:

$$a \mathscr{L} b \quad \text{if and only if} \quad aS = bS. \tag{10}$$

It is easy to see that the relation $\mathscr{L}$ is reflexive, symmetric and transitive, that is, $\mathscr{L}$ is indeed an equivalence relation.[3]) The relation $\mathscr{L}$ determines a decomposition of S into disjoint classes. Denote by L_a the $\mathscr{L}$-class containing the element a in S. We show that L_a is a group.

To show that L_a is a subsemigroup of S, consider the elements a, b in L_a. Then by (9) and (3) we have

$$abS = abS^2 = a(bS)S = a(Sb)S = aS \cap bS = aS = bS,$$

that is, $a \mathscr{L} ab$, and we conclude $ab \in L_a$. Thus L_a is a subsemigroup of S.

Next we show that L_a is a left simple semigroup, i. e. if $b \in L_a$, then $b = ca$,

[2]) It is known that a semigroup S is a disjoint union of groups if and only if it is completely regular (see [3]). Therefore our Theorem 5 implies that *the semigroup S having properties* (2) *and* (3) *concerning left and right ideals respectively, is a completely regular semigroup.*

[3]) See [4]. The relation $\mathscr{L}$ is a two-sidedly stable equivalence relation, that is, $\mathscr{L}$ is a congruence relation on S.

where $c \in L_a$. We know that $ba \in L_a$. Hence $b = xba$, with $x \in S$. Let $xb = c$. To show that $c \in L_a$, let y be an element of S such that $x = yx^2$. Thus

$$b = xba = yx^2ba = (yx)(xba) = yxb = yc.$$

The equations $b = yc$ and $c = xb$ imply $b\mathscr{L}c$, therefore $c \in L_b = L_a$.

Analogously we can prove the right simplicity of the semigroup L_a. Thus the semigroup L_a is both left and right simple, which implies that L_a is a group (see [1]). Hence S is a union of the disjoint classes any of which is a group.

The proof of Theorem 5 is complete.

In what follows we characterize the class of semigroups satisfying either (2) for left ideals or (3) for right ideals.

Theorem 6. *A semigroup S has the property*

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for any two left ideals L_1, L_2 of S if and only if S is left regular and any two left ideals of S commute, i.e. $L_1 L_2 = L_2 L_1$.

Proof. If the semigroup S has the property (2) for left ideals, then by Theorem 3 it is left regular and it follows from (2) that $L_1 L_2 = L_2 L_1$ holds for any left ideals L_1, L_2 of S.

Conversely, suppose that S is a left regular semigroup any two left ideals of which commute. Then $L_1 L_2 \subseteqq L_2$ and $L_1 L_2 = L_2 L_1 \subseteqq L_1$, whence it follows that

$$L_1 L_2 \subseteqq L_1 \cap L_2.$$

To show the converse inclusion, consider an element a of $L_1 \cap L_2$. Then by the left regularity of S we have

$$a = xa^2 = (xa)a \in L_1 L_2.$$

Thus we obtain that

$$L_1 \cap L_2 = L_1 L_2,$$

for any two left ideals of S, which completes the proof.

Similarly, the following result also can be proved.

Theorem 7. *A semigroup S has the property*

$$L_1 \cap L_2 = L_1 L_2$$

for any two left ideals L_1, L_2 of S if and only if S is left regular and each left ideal L of S is a two-sided ideal of S.

Theorems 6 and 7 imply

Theorem 8. *For any semigroup S the following conditions are equivalent:*

(i) *$L_1 \cap L_2 = L_1 L_2$ for any two left ideals L_1, L_2 of S;*

(ii) *S is left regular and $L_1 L_2 = L_2 L_1$ for any two left ideals of S;*

(iii) *S is left regular and each left ideal of S is at the same time a two-sided ideal of S.*

The Theorems 6, 7 and 8 also have a left-right dual. We formulate only the dual of Theorem 8.

Theorem 9. *For any semigroup S the following conditions are equivalent:*

(i) $R_1 \cap R_2 = R_1 R_2$ *for any two right ideals* R_1, R_2 *of* S;

(ii) S *is right regular and* $R_1 R_2 = R_2 R_1$ *for any two right ideals of* S;

(iii) S *is right regular and every right ideal of* S *is a two-sided ideal of* S.

Remark (added in proof). The following example shows that the converse of Theorem 5 is not true:

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	a	b	c	d
d	a	a	d	c

The semigroup S of the elements a, b, c, d with the above multiplication table is a union of the disjoint subgroups $G_1 = \{a, b\}$ and $G_2 = \{c, d\}$, but relation (2) does not hold in S, because G_1 and G_2 are left ideals of S and

$$\emptyset = G_1 \cap G_2 \neq G_1 G_2 = G_2.$$

References

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[4] L. Rédei, *Algebra*. I (Budapest, 1967).

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